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Turán measures

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Abstract

A probability measure σ on the unit circle $\mathbb{T}$ is called a Turán measure if any point of the open unit disc $\mathbb{D}$ is a limit point of zeros of the orthogonal polynomials associated to σ. We show that many classes of measures, including Szegö measures, measures with absolutely convergent series of their parameters, absolutely continuous measures with smooth densities, contain Turán measures.

1. For a probability measure σ on the unit circle $\mathbb{T}$ we denote by $\{\varphi_n\}_{n\geqslant 0}$ the sequence of the orthogonal polynomials in the Hilbert space $L^2(\sigma)$:

$$\varphi_n(z) = k_n(z-\lambda_{1,n})(z-\lambda_{2,n})\dots(z-\lambda_{n,n}), \quad k_n>0,$$

$$\int_{\mathbb{T}} \varphi_i \overline{\varphi_j}\, d\sigma = \begin{cases} 0, & i\neq j, \\ 1, & i=j. \end{cases} \tag{1}$$

It is well-known [10] that $\{\lambda_{1,n}, \lambda_{2,n}, \dots, \lambda_{n,n}\}$ is a subset of the unit disc

$$\mathbb{D} \stackrel{\text{def}}{=} \{z : |z|<1\}.$$

Definition. A probability measure σ on $\mathbb{T}$ is called a Turán measure if for every $\lambda\in\mathbb{D}$ and for every $\varepsilon>0$ there exist positive integers n and k_n, $1\leqslant k_n\leqslant n$, such that

$$|\lambda - \lambda_{k_n,n}|<\varepsilon.$$

In other words, every point of $\mathbb{D}$ is an accumulation point of zeros of the orthogonal polynomials corresponding to σ.

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Theorem (Alfaro-Vigil [1]). *Turán measures exist.*

Any probability measure on $\mathbb{T}$ is uniquely determined by its parameters [3]:

$$a_n = -\frac{\overline{\varphi_{n+1}(0)}}{k_{n+1}}.$$

Putting $z = 0$ in (1), we obtain that

$$a_n = (-1)^n \lambda_{1,n+1}\lambda_{2,n+1}\dots\lambda_{n+1,n+1}. \tag{2}$$

Since all the zeros of nonzero orthogonal polynomials φ_{n+1} in $L^2(d\sigma)$ lie in $\mathbb{D}$, (2) implies that $|a_n| < 1$. If σ is a sum of $n+1$ point masses, then it is natural to consider a polynomial φ_{n+1} identically equal to zero in $L^2(d\sigma)$. Then (2) is valid and $|a_n| = 1$.

From (2) we also obtain that the parameters of any Turán measure must satisfy

$$\liminf_n |a_n| = 0.$$

Another simple necessary condition for a measure σ to be a Turán measure is that the Borel support $\operatorname{supp}(\sigma)$ of σ is $\mathbb{T}$. This follows from the fact that $\{\lambda_{1,n}, \lambda_{2,n}, \dots, \lambda_{n,n}\}$ is a subset of the convex hull of $\operatorname{supp}(\sigma)$.

2. Any probability measure σ with infinite support generates a sequence of the inverse Schur functions (see [5–7]):

$$b_n(z) \stackrel{\text{def}}{=} \frac{\varphi_n(z)}{\varphi_n^*(z)} = \prod_{k=1}^{n} \frac{z - \lambda_{k,n}}{1 - \bar{\lambda}_{k,n} z},$$

where for a polynomial p of degree n we put $p^*(z) \stackrel{\text{def}}{=} z^n \overline{p(1/\bar{z})}$.

Finite Blaschke products b_n satisfy the recurrence

$$b_{n+1}(z) = \frac{z b_n(z) - \bar{a}_n}{1 - z a_n b_n(z)}, \tag{3}$$

which can be easily obtained from Szegö's recurrence (see [10, (11.4.6–11.4.7)] and [5, (7.11)] for details). In what follows $\mathbb{N} \stackrel{\text{def}}{=} \{1, 2, 3, \dots\}$ is the set of positive integers.

Theorem 1. *Given a measure σ with parameters $\{a_n\}_{n\geqslant 0}$ and an arbitrary infinite set $\Lambda \subset \mathbb{N}$ there exists a Turán measure $\sigma^\#$ with parameters $\{a_n^\#\}_{n\geqslant 0}$ satisfying $a_n = a_n^\#$ for $n \notin \Lambda$.*

Proof. It follows the idea of the proof of the Alfaro–Vigil Theorem presented in [9]. We arrange the elements of Λ in an increasing sequence $n_1 < n_2 < \dots < n_k < \cdots$ and take any dense sequence $\{z_k\}_{k\geqslant 1}$ in $\mathbb{D}$. Since by (3) $b_n^\#(z)$ depends only on

$a_0^{\#}, a_1^{\#}, \dots, a_{n-1}^{\#}$, we may define the sequence $a_n^{\#}$ by induction as follows: $a_0^{\#} = a_0$ and

$$a_n^{\#} = \begin{cases} a_n & \text{if } n \neq n_k, \\ \overline{z_k b_{n_k}^{\#}(z_k)} & \text{if } n = n_k \end{cases} \tag{4}$$

for $n > 0$. Then by (3) $b_{n_k+1}^{\#}(z_k) = 0$ and therefore $\sigma^{\#}$ is a Turán measure. □

3. These arguments, however, give no information on metric properties of $\sigma^{\#}$. For instance, is it possible for a Turán measure to be absolutely continuous? To answer this question in positive we need some tools to control $b_{n_k}(z)$. Blaschke products are elements of the unit ball $\mathscr{B}$ of the Hardy algebra in the unit disc:

$$\mathscr{B} = \{f(z): f \text{ is holomorphic in } \mathbb{D},\ |f(z)| \leqslant 1 \text{ for } z \in \mathbb{D}\}.$$

Schur's algorithm

$$f(z) \stackrel{\text{def}}{=} f_0(z) = \frac{zf_1(z) + f_0(0)}{1 + \overline{f_0(0)}zf_1(z)}; \dots; f_n(z) = \frac{zf_{n+1} + f_n(0)}{1 + \overline{f_n(0)}zf_{n+1}(z)}; \dots$$

determines a one-to-one mapping

$$\mathscr{S} : \mathscr{B} \to \mathscr{S}^{\infty},$$

$$\mathscr{S}(f) = (f_0(0), f_1(0), f_2(0), \dots)$$

of $\mathscr{B}$ onto the space of parameters $\mathscr{S}^{\infty}$. This mapping is a homeomorphism of the topological spaces $\mathscr{B}$ and $\mathscr{S}^{\infty}$ equipped with the topology of pointwise convergence (see [5, Lemma 4.11; 6, Theorem 1.1] for details).

Definition. A probability measure σ with parameters $\{a_n\}_{n \geqslant 0}$ is called a Markoff measure if there exist $\varepsilon > 0$ and a positive integer l such that

$$\max_{j=0,1,\dots,l} |a_{n+j}| > \varepsilon \quad \text{for } n = 0, 1, \dots\ .$$

If σ is not a Markoff measure, then there exists an infinite set Λ of the form

$$\Lambda = \bigcup_{k=1}^{\infty} [m_k - l_k, m_k],$$

$$m_k < m_{k+1} - l_{k+1}, \quad k = 1, 2, \dots, \quad \lim_k l_k = +\infty, \tag{5}$$

such that

$$\lim_{n \in \Lambda} a_n = 0. \tag{6}$$

Any subset Λ of $\mathbb{N}$ can be written in the form $\Lambda = \{n_1, n_2, n_3, \dots, n_k, \dots\}$, where $n_1 < n_2 < n_3 < \cdots < n_k < \cdots$.

Theorem 2. *If σ is not a Markoff measure, $\{\varepsilon_k\}_{k \geqslant 1}$ is an arbitrary positive sequence, then there exist an infinite subset $\Lambda = \{n_1, n_2, \dots, n_k, \dots\}$ in $\mathbb{N}$ satisfying $n_{k+1}/n_k > \varepsilon_k^{-1}$,* *and a Turán measure* $\sigma^{\#}$ *such that the parameters* a_n *of* σ *and* $a_n^{\#}$ *of* $\sigma^{\#}$ *are equal for* $n \notin \Lambda$ *and* $|a_{n_k}^{\#}| < \varepsilon_k$ *for* $k = 1, 2, 3, \ldots$.

Proof. It follows from (5) to (6) and the following lemma.

Lemma 3. *Let* σ *be a probability measure on* $\mathbb{T}$ *with parameters* $\{a_n\}_{n \geqslant 0}$ *and* Λ *be a subset in* $\mathbb{N}$ *of the form* (5) *such that* (6) *holds. Then for every sequence* $\{\varepsilon_j\}_{j \geqslant 1}$ *of positive numbers there is an infinite subset* $\Lambda_0 = \{m_{k_1}, m_{k_2}, \ldots, m_{k_j}, \ldots\}$ *of* Λ *with* $m_{k_{j+1}}/m_{k_j} > \varepsilon_j^{-1}$, *and a Turán measure* $\sigma^{\#}$ *such that the parameters* a_n *of* σ *and* $a_n^{\#}$ *of* $\sigma^{\#}$ *are equal for* $n \notin \Lambda_0$ *and* $|a_{m_{k_j}}^{\#}| < \varepsilon_j$ *for* $j = 1, 2, 3, \ldots$.

Proof. It follows from (3) that

$$\mathscr{S} b_{m_k} = (-\overline{a_{m_k - 1}}, -\overline{a_{m_k - 2}}, \ldots, 1).$$

Since $\mathscr{S}$ is a homeomorphism, (5) and (6) imply that

$$b_{m_k}(z) \rightrightarrows 0 \tag{7}$$

uniformly on compact subsets of $\mathbb{D}$.

Let $\{z_k\}_{k \geqslant 1}$ be a dense sequence in $\mathbb{D}$. By (7) there is $j_1 > 1$ such that

$$|b_{m_{j_1}}(z_1)| < \varepsilon_1.$$

We put

$$a_n^{(1)} = \begin{cases} a_n & \text{if } n \neq m_{j_1}, \\ \overline{z_1 b_{m_{j_1}}(z_1)} & \text{if } n = m_{j_1} \end{cases}$$

and consider the measure $\sigma^{(1)}$ corresponding to the parameters $\{a_n^{(1)}\}_{n \geqslant 0}$. Since

$$\lim_{n \in \Lambda} a_n^{(1)} = 0,$$

we obtain that there is $j_2 > j_1$ such that

$$m_{j_2} > \varepsilon_1^{-1} m_{j_1},$$

$$|b_{m_{j_2}}^{(1)}(z_2)| < \varepsilon_2.$$

We put

$$a_n^{(2)} = \begin{cases} a_n^{(1)} & \text{if } n \neq m_{j_2}, \\ \overline{z_2 b_{m_{j_2}}^{(1)}(z_2)} & \text{if } n = m_{j_2} \end{cases}$$

and consider the measure $\sigma^{(2)}$ corresponding to the parameters $\{a_n^{(2)}\}_{n \geqslant 0}$.

This construction can be continued by induction. Then for any integer k, $k > 1$, we obtain: a finite subset $m_{j_1} < m_{j_2} < \cdots < m_{j_k}$ of $\mathbb{N}$ satisfying

$$m_{j_{s+1}} > \varepsilon_s^{-1} m_{j_s}, \quad s = 1, 2, \ldots, k - 1; \tag{8a}$$

and a finite family of parameters $\{a_n^{(1)}\}_{n\geqslant 0}$, $\{a_n^{(2)}\}_{n\geqslant 0}$, ..., $\{a_n^{(k)}\}_{n\geqslant 0}$ such that

$$a_n^{(s+1)} = \begin{cases} a_n^{(s)} & \text{if } n \neq m_{j_{s+1}}, \\ \dfrac{}{z_{s+1} b_{m_{j_{s+1}}}^{(s)}(z_{s+1})} & \text{if } n = m_{j_{s+1}}, \end{cases} \quad s = 1, 2, \dots, k-1 \tag{8b}$$

$$|a_{m_{j_{s+1}}}^{(s+1)}| < \varepsilon_{s+1}, \quad s = 1, 2, \dots, k-1. \tag{8c}$$

We denote by $\sigma^{(j)}$ the measure corresponding to the parameters $\{a_n^{(j)}\}_{n\geqslant 0}$. By (8b) the first parameters $a_0^{\#}, a_1^{\#}, \dots, a_{m_{k_j}}^{\#}$ of $\sigma^{(j)}$, $\sigma^{(j+1)}$, $\sigma^{(j+2)}$, ... are the same. It follows that the limit $*\text{-}\lim_j \sigma^{(j)}$ exists in the $*$-weak topology of the space of probability measures. Moreover, the limit measure $\sigma^{\#}$ has the parameters $\{a_n^{\#}\}_{n\geqslant 0}$. By (8b) and (3) its orthogonal polynomials satisfy $\varphi_{m_{j_s}+1}^{\#}(z_{j_s}) = 0$, $s = 1, 2, 3, \dots$. Finally, (8b) and (8c) imply $a_n = a_n^{\#}$ for $n \neq m_{j_s}$ and also $|a_n^{\#}| < \varepsilon_j$ if $n = m_{j_s}$. □

Definition. We say that a probability measure σ is in $\mathcal{M}_p$, $0 < p < +\infty$, if its parameters $\{a_n\}_{n\geqslant 0}$ satisfy

$$\sum_{n=0}^{\infty} |a_n|^p < \infty.$$

We say that σ is in Nevai's class $\mathcal{M}_\infty$ if $\lim_n a_n = 0$.

Theorem 4. *For any $\sigma \in \mathcal{M}_p$, $0 < p \leqslant +\infty$ and any infinite subset Λ of $\mathbb{N}$ there exist an infinite set $\Lambda_0 \subset \Lambda$ and a Turán measure $\sigma^{\#}$ in $\mathcal{M}_p$ such that the parameters a_n of σ and $a_n^{\#}$ of $\sigma^{\#}$ are equal for $n \notin \Lambda_0$.*

Proof. Since $\mathcal{M}_p \subset \mathcal{M}_\infty$, there exist a subsequence $\{m_k\}_{k\geqslant 1}$ in Λ and a sequence $\{l_k\}_{k\geqslant 1}$ in $\mathbb{N}$ such that the set (5) corresponding to these sequences satisfies (6). Choosing now $\varepsilon_j = 2^{-j}$, $j = 1, 2, \dots$, we complete the proof by Lemma 3. □

Remark. By Geronimus theorem [3, Theorem 8.2] σ is a Szegö measure if and only if $\sigma \in \mathcal{M}_2$. Hence by Theorem 4 any Szegö measure generates infinitely many Turán measures with parameters almost identical to the parameters of the initial Szegö measure. By another Geronimus theorem [3, Theorem 8.5] any measure in $\mathcal{M}_1$ is absolutely continuous, which together with Theorem 4 imply the existence of absolutely continuous Turán measures with continuous density.

Theorem 5. *If σ is a Markoff measure, then for every infinite set $\Lambda = \{n_1, n_2, \dots, n_k, \dots\}$ with $\lim_k (n_{k+1} - n_k) = \infty$ there exists a Turán measure $\sigma^{\#}$ in Markoff's class such that the parameters a_n of σ and $a_n^{\#}$ of $\sigma^{\#}$ are equal for $n \notin \Lambda$.*

Proof. Let l be the integer in the definition of a Markoff measure for σ. Then in any segment $[j \cdot 2l, (j+1) \cdot 2l]$ there are at least two indexes i such that $|a_i| > \varepsilon > 0$. Since

$\lim_k (n_{k+1} - n_k) = \infty$, every segment $[j \cdot 2l, (j+1) \cdot 2l]$ with j sufficiently big may contain at most one element of Λ. Applying Theorem 1 to σ and Λ, we obtain the result. $\square$

It was shown in [7, Theorems 1.8 and G], that almost all zeros of the orthogonal polynomials associated to a Markoff measure accumulate to supp (σ). At the same time every disc $\{z : |z| < r\}$, $0 < r < 1$, can contain only $\mathrm{Const} \cdot (1-r)^{-1}$ zeros of $\varphi_n(z)$. Theorem 5 shows that these zeros can accumulate to any point of the disc $\{|z| < r\}$.

4. Is it possible for a Turán measure to be absolutely continuous with infinitely differentiable density?

Let

$$\Phi_n(z) = \frac{\varphi_n(z)}{k_n}, \quad n = 0, 1, 2, \ldots$$

be the sequence of *monic* orthogonal polynomials associated to a probability measure σ.

For any Szegö measure σ we define the Szegö function by

$$D(z, \sigma) = \exp\left(\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \log \sqrt{\sigma'}\, dm(\zeta)\right),$$

where m is the normalized $(m(\mathbb{T}) = 1)$ Lebesgue measure on $\mathbb{T}$. Szegö's Theorem says that

$$\Phi_n^*(z) \rightrightarrows D(z, \sigma)^{-1} \tag{9}$$

uniformly on compact subsets of $\mathbb{D}$ for any Szegö measure σ.

For $0 < r < 1$ we denote by $\mathscr{E}_r(\mathbb{T})$ the set of probability measures on $\mathbb{T}$ with parameters $\{a_n\}_{n \geqslant 0}$ satisfying $a_n = o(r^n)$. Since

$$\sum_{n=0}^{\infty} |a_n|^2 < \sum_{n=0}^{\infty} |a_n| < \mathrm{Const} \sum_{n=0}^{\infty} r^n < \infty,$$

every measure in $\mathscr{E}_r(\mathbb{T})$ is a Szegö measure. Next we put

$$\mathscr{E}_0(\mathbb{T}) = \bigcap_{0<r<1} \mathscr{E}_r(\mathbb{T}), \qquad \mathscr{E}_1(\mathbb{T}) = \bigcup_{0<r<1} \mathscr{E}_r(\mathbb{T}).$$

Theorem (Nevai–Totik [8]). *A measure σ is in $\mathscr{E}_0(\mathbb{T})$ if and only if $D(z,\sigma)^{-1}$ extends to an entire function with zeros $\{w_k\}_{k \geqslant 1}$ in $\mathbb{C} \setminus \mathbb{D}$. The zeros of orthogonal polynomials associated to σ accumulate either to 0 or to the points $1/\bar{w}_k$, $k = 1, 2, \ldots$.*

By the Nevai–Totik Theorem there are no Turán measures in $\mathscr{E}_0(\mathbb{T})$. Moreover, it is proved in [8] that there are no Turán measures in $\mathscr{E}_1(\mathbb{T})$. The relation between the rate of decay of the parameters and the size of the largest disk, wherein Φ_n^* are bounded, found in [8] imply that zeros of the orthogonal polynomials of such measures cannot accumulate to any point in $\mathbb{D}$. In the proof of Theorem 6 we partially reproduce arguments of [8] to obtain formulas (11a) and (11b).

For a positive sequence $\{\varepsilon_n\}_{n\geqslant 0}$ we denote by $\mathscr{P}(\{\varepsilon_n\}_{n\geqslant 0})$ the set of probability measures σ such that their parameters $\{a_n\}_{n\geqslant 0}$ satisfy

$$|a_n| < C_\sigma \cdot \varepsilon_n, \quad n = 0, 1, 2, \ldots, \quad C_\sigma > 0.$$

Theorem 6. *Let $\{\varepsilon_n\}_{n\geqslant 0}$ be a positive sequence such that for every $t, 0 < t < 1$,*

$$t^n = o(\varepsilon_n). \tag{10}$$

Then for every $\sigma \in \mathscr{E}_1(\mathbb{T})$, with parameters $\{a_n\}_{n\geqslant 0}$ and for an arbitrary infinite set $\Lambda \subset \mathbb{N}$ there is a Turán measure $\sigma^{\#}$ in $\mathscr{P}(\{\varepsilon_n\}_{n\geqslant 0})$ with parameters $\{a_n^{\#}\}_{n\geqslant 0}$ such that $a_n = a_n^{\#}$ for $n \notin \Lambda$.

Remark. Condition (10) is equivalent to the inclusion

$$\mathscr{E}_1(\mathbb{T}) \subset \mathscr{P}(\{\varepsilon_n\}_{n\geqslant 0}).$$

Proof. If $\sigma \in \mathscr{E}_1(\mathbb{T})$, then there is $0 < r < 1$ such that $\sigma \in \mathscr{E}_r(\mathbb{T})$. Using the monic orthogonal polynomials, we can rewrite Szegö's recurrence formulas as follows:

$$\begin{pmatrix} z^{-(n+1)} \cdot \Phi_{n+1}(z) \\ \Phi_{n+1}^*(z) \end{pmatrix} = \begin{pmatrix} 1 & -\bar{a}_n z^{-(n+1)} \\ -a_n z^{(n+1)} & 1 \end{pmatrix} \cdot \begin{pmatrix} z^{-n} \cdot \Phi_n(z) \\ \Phi_n^*(z) \end{pmatrix},$$

where

$$\begin{pmatrix} 1 & -\bar{a}_n z^{-(n+1)} \\ -a_n z^{(n+1)} & 1 \end{pmatrix} = I + \begin{pmatrix} 0 & -\bar{a}_n z^{-(n+1)} \\ -a_n z^{(n+1)} & 0 \end{pmatrix} \stackrel{\text{def}}{=} I + A_n.$$

It is clear that the matrix norm of $I + A_n$ satisfies

$$||I + A_n|| \leqslant 1 + ||A_n|| = 1 + |a_n| \cdot (\max(|z|, |z|^{-1}))^{n+1}.$$

It follows that for every $z \neq 0$

$$\sqrt{|z^{-(n+1)}\Phi_{n+1}(z)|^2 + |\Phi_{n+1}^*(z)|^2} \leqslant \sqrt{2} \prod_{k=0}^{n} ||I + A_k|| \leqslant \sqrt{2} \prod_{k=0}^{n} (1 + ||A_k||)$$
$$\leqslant \sqrt{2} \exp\left\{ \sum_{k=0}^{n} |a_k| (\max(|z|, |z|^{-1}))^{k+1} \right\}.$$

Assuming that $0 < r < \delta \leqslant |z| \leqslant 1/\delta$, we obtain that

$$|\Phi_{n+1}^*(z)| \leqslant C_\delta, \tag{11a}$$

$$|\Phi_{n+1}(z)| \leqslant C_\delta |z|^{n+1}, \tag{11b}$$

since

$$\sum_{k=0}^{\infty} |a_k| \left(\frac{1}{\delta}\right)^{k+1} < \text{Const} \cdot \sum_{k=0}^{\infty} \left(\frac{r}{\delta}\right)^{k+1} < \infty.$$

It follows from (11a) that the sequence of polynomials $\Phi_n^*(z)$ is normal in the disc $\{z : |z| < 1/\delta\}$. Now (9) implies that $D(z,\sigma)^{-1}$ extends to a holomorphic function in $\{z : |z| < 1/\delta\}$, which cannot have zeros in $\mathbb{D}$ because $D(z,\sigma)^{-1}$ does not vanish in $\mathbb{D}$. Therefore for every $0 < \gamma < 1$ the family of holomorphic functions $1/\Phi_n^*(z)$ is uniformly bounded in the disc $\{z : |z| < \gamma\}$. This together with (11b) imply that for every γ satisfying $\delta < \gamma < 1$ there is a positive constant C_γ such that

$$|b_n(z)| = \left|\frac{\Phi_n(z)}{\Phi_n^*(z)}\right| < C_\gamma |z|^n \tag{12}$$

for $\delta < |z| < \gamma$.

We consider now any dense sequence $\{z_j\}_{j\geqslant 1}$ in $\mathbb{D}$. For $|z_1| < \gamma_1 < t_1 < 1$ we obtain from (10) that

$$C_{\gamma_1}\gamma_1^n < t_1^n < \varepsilon_n$$

for $n \geqslant n_1 \in \Lambda$. It follows from (12) that $|b_n(z)| < \varepsilon_n$ for $n = n_1$ and $\delta < |z| < \gamma_1$. Since b_n is holomorphic in $\mathbb{D}$, by the maximum modulus theorem we conclude that

$$|b_{n_1}(z_1)| < \varepsilon_{n_1}.$$

Modifying $\{a_n\}_{n\geqslant 0}$ by (4), we obtain a new element $\sigma^{(1)}$ in $\mathscr{E}_r(\mathbb{T})$ with parameters $\{a_n^{(1)}\}_{n\geqslant 0}$.

Applying the above arguments to $\sigma^{(1)}$ and Λ, we find an integer $n_2 \in \Lambda$, $n_1 < n_2$, such that

$$|b_{n_2}^{(1)}(z_2)| < \varepsilon_{n_2}.$$

We modify the parameters $\{a_n^{(1)}\}_{n\geqslant 0}$ by (4) and obtain a new element $\sigma^{(2)}$ in $\mathscr{E}_r(\mathbb{T})$ with parameters $\{a_n^{(2)}\}_{n\geqslant 0}$. Running the above construction by induction, we obtain a sequence of measures $\sigma^{(k)}$ in $\mathscr{E}_r(\mathbb{T})$. The first parameters $a_0^{\#}, a_1^{\#}, \dots, a_{n_j}^{\#}$ of the measures $\sigma^{(j)}$, $\sigma^{(j+1)}$, $\sigma^{(j+2)}$, ... are the same. It follows that the limit $*\text{-}\lim_j \sigma^{(j)}$ exists, the parameters of $\sigma^{\#}$ are $\{a_n^{\#}\}_{n\geqslant 0}$, and $b_{n_k}^{(k-1)} = b_{n_k}^{\#}$. Since

$$z_k b_{n_k}^{\#}(z_k) = \overline{a_{n_k}^{\#}}$$

by the construction, we obtain from (3) that $b_{n_k+1}^{\#}(z_k) = 0$, which implies that

$$\varphi_{n_k+1}^{\#}(z_k) = 0, \quad k = 1, 2, \dots .$$

Since $\{z_k\}_{k\geqslant 1}$ is dense in $\mathbb{D}$, we see that $\sigma^{\#}$ is a Turán measure. The definition of $a^{\#}_{n_k}$ implies that

$$|a^{\#}_{n_k}|<|b^{(k-1)}_{n_k}(z_k)|<\varepsilon_{n_k}.$$

Since $\sigma\in\mathscr{E}_r(\mathbb{T})$ and $a^{\#}_n=a_n$ if $n\notin\Lambda$, (10) implies that $\sigma^{\#}\in\mathscr{P}(\{\varepsilon_n\}_{n\geqslant 0})$. $\square$

Remark. Although by Nevai and Totik [8] there are no Turán measure in $\mathscr{E}_1(\mathbb{T})$, Theorem 6 says that any measure in $\mathscr{E}_1(\mathbb{T})$ generates infinitely many Turán measures with almost the same parameters in $\mathscr{P}(\{\varepsilon_n\}_{n\geqslant 0})$, which may be arbitrary close to $\mathscr{E}_1(\mathbb{T})$.

Corollary 7. *There exists a Turán measure with infinitely differentiable density.*

Proof. Let $\{\varepsilon_n\}_{n\geqslant 0}$ be a sequence such that $\varepsilon_n=o(n^{-p})$ for every $p>0$. For instance we may put $\varepsilon_n=\exp\{-n^{\alpha}\}$, $n=0,1,2,\dots,\alpha>0$. Then every σ in $\mathscr{P}(\{\varepsilon_n\}_{n\geqslant 0})$ is absolutely continuous and $1/\sigma'$ is infinitely differentiable. This result was first obtained in [2] (see [4,6, Theorem 11] for another proofs). If σ is the measure with parameters $a_n=\exp\{-(n+1)^2\}$, $n=0,1,2,\dots,$ then $\sigma\in\mathscr{E}_0(\mathbb{T})$. The result now follows by Theorem 6 applied to σ, $\{\varepsilon_n\}_{n\geqslant 0}$ and $\Lambda=\{2^{2^n}\}_{n\geqslant 0}$. $\square$

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